Homework 7 Solutions

1. On the 10/24 Recitation Worksheet, you showed that the 2×2 determinant formula satisfies the first two determinant axioms. Now show the formula satisfies the third determinant axiom (linearity in each row).

   We will show linearity in the first row; showing it for the second row is similar.

   - $\det \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c & d \end{bmatrix} = (a_1 + a_2)d - (b_1 + b_2)c = a_1 d - b_1 c + a_2 d - b_2 c$
     $= \det \begin{bmatrix} a_1 & b_1 \\ c & d \end{bmatrix} + \det \begin{bmatrix} a_2 & b_2 \\ c & d \end{bmatrix}$
   - $\det \begin{bmatrix} ka & kb \\ c & d \end{bmatrix} = (ka)d - (kb)c = k(ad - bc) = k \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

2. Recall that if A is a 2×2 matrix, then $|\det A|$ is the area of the parallelogram formed by the columns of A and if A is 3×3, the $|\det A|$ is the volume of the parallelopiped formed by the columns of A. More generally, if A is $n \times n$, where $n \geq 4$, then $|\det A|$ is called the "hypervolume" of the $n-$dimensional region formed by the columns of A.

   (a) Use determinants to find the area of the triangle with corners at the coordinate points $(3, 4)$, $(2, 1)$, and $(0, 0)$.

   Let $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$. Then the area of the parallelogram formed by the columns of A is $|\det A| = 5$. Now, the triangle in question comprises half of the parallelogram; hence its area is $\frac{5}{2}$.

   (b) Consider the $4-$dimensional region H given by the vectors

   $$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

   Use determinants to find the hypervolume of H. Explain your steps (you use cofactor expansions or row reduction).

We will use row reduction.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 \end{vmatrix} \text{ (Replace R}i\text{ with R}i\text{-R1 for } i = 2, 3, 4)$$

$$= - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \end{vmatrix} \text{ (Swap R2 and R4)}$$

$$= - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & -2 & -2 \end{vmatrix} \text{ (Replace R3 by R3-R2)}$$

$$= - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & -4 \end{vmatrix} \text{ (Replace R4 by R4-R3)}$$

$$= -(1)(-2)(-2)(-4) = 16$$

Since the determinant of the matrix whose columns are the vectors forming H is 16, the volume of H is 16.

(c) Show that the vectors in part (b) are orthogonal and have the same length. Such an object is called a *hypercube* (a 4-dimensional analog of a cube (with is a 3-dimensional analog of a square)). Analogous to the area formula of a square or the volume formula of a cube, the volume of a hypercube with side length s is given by s^4; use this formula to verify your answer to part (b).

We show they are orthogonal by showing the dot product of each pair is zero:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = 0, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0,$$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0, \quad \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = 0$$

Since $\sqrt{(\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2} = 2$, the length of each of vector is 2. The base of the hypercube is a regular 3-dimensional cube of side length 2, which has

volume $2^3 = 8$. Now the height of the hypercube has length 2; hence the hypervolume is $2 \cdot 8 = 16$, which agrees with part (b).

3. Use cofactor expansions to find the determinant of the following matrix. Be sure to show your work and explain your steps.

$$A = \begin{bmatrix} -2 & 1 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 1 & -2 & 4 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

$$\begin{vmatrix} -2 & 1 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 1 & -2 & 4 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 1(-1)^5 \begin{vmatrix} 0 & 3 & 1 \\ 1 & -2 & 4 \\ 1 & 1 & 1 \end{vmatrix} + 1(-1)^8 \begin{vmatrix} -2 & 1 & 2 \\ 0 & 3 & 1 \\ 1 & -2 & 4 \end{vmatrix} \text{ (C4 Cofactor Expansion)}$$

$$= -\left(3(-1)^3 \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} + 1(-1)^4 \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix}\right) \text{ (R1 Cofactor Expansion)}$$

$$+ \left(-2(-1)^2 \begin{vmatrix} 3 & 1 \\ -2 & 4 \end{vmatrix} + 1(-1)^4 \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}\right) \text{ (C1 Cofactor Expansion)}$$

$$= -(-3(-3) + (1)(3)) + (-2(14) + 1(-5))$$

$$= -45$$

4. The cofactor formula of the inverse is not typically used for computational purposes (as it is much less efficient than row reduction). Rather, it is useful for proving general facts about inverses. Use the cofactor formula for inverses to prove that if A is an invertible 3×3 upper triangular matrix, then A^{-1} is also upper triangular.

[Note: this fact can similarly be proved to be true for any $n \times n$ matrix]

Since $A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$, we need to show that $C_{12} = C_{13} = C_{23} = 0$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$. Then

$$C_{12} = (-1)^3 \begin{vmatrix} 0 & a_{23} \\ 0 & a_{33} \end{vmatrix} = 0$$

$$C_{13} = (-1)^4 \begin{vmatrix} 0 & a_{22} \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{23} = (-1)^5 \begin{vmatrix} a_{11} & a_{12} \\ 0 & 0 \end{vmatrix} = 0$$

5. Let $Q_{\frac{\pi}{4}}$ and $R_{\frac{\pi}{4}}$ be the 2×2 rotation and reflection matrices defined on Homework 6. Determine the eigenvalues of these two matrices geometrically; if there are no eigenvalues, explain why. You shouldn't need to compute anything for this problem.

   $Q_{\frac{\pi}{4}}$ rotates vectors in $\mathbb{R}^2$ by an angle of $\frac{\pi}{4}$. It follows that for any nonzero vector $\mathbf{v}$, $Q_{\frac{\pi}{4}}\mathbf{v}$ can never be a scalar multiple of $\mathbf{v}$; hence there are not (real) eigenvalues or eigenvectors.

   $R_{\frac{\pi}{4}}$ reflects vectors in $\mathbb{R}^2$ over the line $y = x$. Since the line $y = x$ is fixed pointwise by this reflection, if $\mathbf{v}$ is in this line, then $R_{\frac{\pi}{4}}\mathbf{v} = \mathbf{v}$ and so $\lambda = 1$ is an eigenvalue. Since the line $y = -x$ is send to itself under the reflection, $\mathbf{v}$ is in this line, then $R_{\frac{\pi}{4}}\mathbf{v} = -\mathbf{v}$ and so $\lambda = -1$ is an eigenvalue. Since this matrix is 2×2 there can be at most two eigenvalues, and so 1 and -1 are the only eigenvalues.

6. Let P be an $n \times n$ projection matrix that projects vectors onto a subspace W of $\mathbb{R}^n$.

   (a) Algebraically prove that if λ is an eigenvalue of P then $\lambda \in \{0, 1\}$ (Hint: recall from Homework 5 that $P^2 = P$).

   Let $\mathbf{v}$ be an eigenvector of P with corresponding eigenvalue λ; then $P\mathbf{v} = \lambda\mathbf{v}$. Hence $P^2\mathbf{v} = \lambda^2\mathbf{v}$. Since $P^2 = P$, we thus have $P\mathbf{v} = \lambda^2\mathbf{v}$; so $\lambda^2\mathbf{v} = \lambda\mathbf{v}$. Since $\mathbf{v}$ is nonzero, we have $\lambda^2 = \lambda$ or $\lambda^2 - \lambda = 0$, which implies $\lambda = 0, 1$.

   (b) Suppose $0 < \dim W < n$. Geometrically show that P has both eigenvalues 0 and 1. Determine the corresponding eigenspaces.

   Since $\dim W > 0$, W contains nonzero vectors. Let $\mathbf{v} \in W$ be nonzero. Then $P\mathbf{v} = \mathbf{v}$ and so $\mathbf{v}$ is an eigenvector with eigenvalue 1. Moreover, if $\mathbf{v} \notin W$, then $P\mathbf{v} \neq \mathbf{v}$; hence $E_1(A) = W$. Since $\dim W < n$, then $W^\perp \neq \{\mathbf{0}\}$ and so $W^\perp$ contains nozero vectors. Let $\mathbf{w} \in W^\perp$ be nonzero. Then $P\mathbf{w} = \mathbf{0}$ and so $\mathbf{w}$ is an eigenvector with eigenvalue 0. Moreover, if $\mathbf{v} \notin W^\perp$, then $P\mathbf{v} \neq \mathbf{0}$; hence $E_0(A) = W^\perp$.

7. Consider the Markov matrix $M = \begin{bmatrix} .2 & 0 & 0 \\ 0 & .2 & 0 \\ .8 & .8 & 1 \end{bmatrix}$.
   Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{x}_0 = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$.

   (a) Show that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are eigenvectors of M and find the corresponding eigenvalues.

Since $M\mathbf{v}_1 = \begin{bmatrix} .2 & 0 & 0 \\ 0 & .2 & 0 \\ .8 & .8 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1\mathbf{v}_1$, $\mathbf{v}_1$ is an eigenvector with corresponding eigenvalue 1.

Since $M\mathbf{v}_2 = \begin{bmatrix} .2 & 0 & 0 \\ 0 & .2 & 0 \\ .8 & .8 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -.2 \\ .2 \\ 0 \end{bmatrix} = .2\mathbf{v}_2$, $\mathbf{v}_2$ is an eigenvector with corresponding eigenvalue .2.

Since $M\mathbf{v}_2 = \begin{bmatrix} .2 & 0 & 0 \\ 0 & .2 & 0 \\ .8 & .8 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -.2 \\ 0 \\ .2 \end{bmatrix} = .2\mathbf{v}_3$, $\mathbf{v}_3$ is an eigenvector with corresponding eigenvalue .2.

(b) Set up and solve a system in order to express $\mathbf{x}_0$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

We need to find scalars c_1, c_2, c_3 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{x}_0$. This corresponds to solving the system $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \mathbf{x}_0$.

Reducing the augmented matrix $\left[\begin{array}{ccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{x}_0 \end{array}\right]$ yields $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & .6 \\ 0 & 0 & 1 & -.9 \end{array}\right]$.

Hence $\mathbf{x}_0 = \mathbf{v}_1 + .6\mathbf{v}_2 - .9\mathbf{v}_3$.

(c) Use part (b) to determine a formula for $\mathbf{x}_k = M^k\mathbf{x_0}$ in terms of k.

$$\begin{aligned} \mathbf{x}_k = M^k\mathbf{x_0} &= M^K(\mathbf{v}_1 + .6\mathbf{v}_2 - .9\mathbf{v}_3) \\ &= M^k\mathbf{v}_1 + .6M^k\mathbf{v}_2 - .9M^k\mathbf{v}_3 \\ &= (1)^k\mathbf{v}_1 + .6(.2)^k\mathbf{v}_2 - .9(.2)^k\mathbf{v}_3 \\ &= \mathbf{v}_1 + .6(.2)^k\mathbf{v}_2 - .9(.2)^k\mathbf{v}_3 \end{aligned}$$

(d) Determine what the Markov chain defined in part (c) converges to by computing its limit.

As $k \to \infty$, $(.2)^k \to 0$. Hence $\mathbf{x}_k \to \mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.